Lepton Flavour Violation

Sacha Davidson
IPN de Lyon/CNRS, France

0. Preliminaries
   - What is LFV?
   - Why is it interesting: flavour conserved in the SM
   - Some phenomenology + kinematics
   - This course: some models → EFT
1. Meet the vector/scalar leptoquark! (induces "low energy" LFV at tree level)
   - at $p^2 \ll m_{LQ}^2$, induces 2-quark 2-lepton interactions at tree level
   - Fierz transformations
   - dimension 6 "effective operators"
   - constraints from the PDB*lattice
2. (the type II seesaw: add a heavy scalar with LNV interactions (more low energy LFV at tree level))
   - NP that induces majorana m_ν at tree level
   - and 4-lepton LFV contact interactions
   - constraints
3. Dirac neutrino masses: new *light* physics and LFV at loop
   - add singlet fermions("RH neutrinos") to SM, and conserve lepton number
   - Dirac m_ν (like quarks) and PMNS matrix
   - FCNC at loop, suppressed by GIM mechanism
   - caltimate $\mu \to e\gamma$, $\mu \to e\bar{e}e$ in SM...
   - **...and identify as a dimension 5 effective operator in QED!**
     "integrating out" the $W(Z, h)$, in loops"

4. the type I seesaw

   - add singlet fermions, and allow large lepton number violating masses M_I
   - at low energies $\ll M_I$, doublet neutrinos have LNV ("majorana") masses from a dim5 effective operator
   - to calculate $\mu \to e\gamma$ in the type I seesaw
     - in the full theory: heavy and light neutrinos in the loop
     - in an effective theory when $v \ll M_I$: remove the heavy neutrino, replace by
       (a) the $\mu \to e\gamma$ contact interaction induced by the heavy neutrino
       (b) a dim six contact interaction giving a non-unitarity contribution to light neutrino propagation
       (c) the light neutrino mass matrix

5. loops, dim reg, continuum EFT a la Georgi and no form factors

   - renormalisation à la Wilson: theories travel in parameter space as a function of scale
   - but we regularise in dim reg (and renormalise in $\overline{MS}$):can we put the physics of scales back by hand?

   $\Rightarrow$ EFT à la Georgi —recipe:
   – use dim reg (scale μ)
   – attribute physical significance to μ by removing particles from the theory when μ drops below their mass
   – match (equate) Greens functions, at $\mu = m$, of theories valid above and below μ
   (theory below m will contain *local* contact interactions to reproduce effects of particles of mass $> m$)
   – run, using RGs each theory in its range of μ (this puts correct threshholds in logs)

6. Operator bases and equations of motion

   -

Lepton Flavour Violation—

1. LFV ≡ contact interaction involving charged leptons, changing the flavour of the charged leptons (will define a contact interaction later)
   - not include neutrino oscillations (not contact interaction)
   - equivalent to FCNC in the down-type quark sector
2. LFV is interesting because lepton flavour is conserved in the SM Lagrangian (with massless neutrinos): charged lepton mass matrix selects an eigenbasis in flavour space; no other interaction does.
   - $\mathcal{L}$ contains flavour diagonal kinetic terms (for charged/neutral leptons)
   - so can work in charged lepton mass basis
   - at W vertex, label neutrino by the flavour of charged lepton

   ⇒ no flavour change.
3. LFV is neccessairly present once include neutrino masses! (see chapter 4).
   - kinetic terms still flavour diagonal
   - put charged *and neutral* leptons in their mass bases
   - charged and neutral mass eigenstates meet at W vertex: mixing angles and flavour change...
   - **But maybe NP which is irrelevant to m_ν could give LFV interactions?**
     Maybe the neutrino mass generation mechanism gives other LFV interactions?

0.1 Conventions, fermions, etc

γ matrices

Use Peskin conventions, and notation. (NB: Wess+Bagger use metric $(-,+++)$.) Take

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{0.1}$$

and

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.2}$$

where $\bar{\sigma}^\mu = (\sigma^0, -\sigma^i)$.
So $\gamma^{0\dagger} = \gamma^0$, and $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$.

spinors

I mostly 4-component spinors ψ; chiral decomposition $\psi = \psi_L + \psi_R$,

$$\psi_L = P_L\,\psi \quad \text{avec} \quad P_L = \frac{(1-\gamma_5)}{2} \quad , \qquad \psi_R = P_R\,\psi \quad \text{avec} \quad P_R = \frac{(1+\gamma_5)}{2} \quad , \qquad P_L^2 = P_L, \quad P_R^2 = P_R$$

chirality is *not* an observable ($\rightarrow$ helicity $= \pm\hat{s}\cdot\hat{k} = \pm 1/2$ in relativistic limit), but $P_{L,R}$ simple to calculate with :)

Careful about notation: $\overline{(\psi_R)} = (\overline{\psi})_L \neq (\overline{\psi})_R$

Fermions follow fermi statistics, so something anti-commutes. In the second quantised formalism of P+S ($\hat{\psi} \sim \int(\hat{a}u + \hat{b}^\dagger v)$, see eg P+S 3.99), the $\hat{a}, \hat{b}$ operators anti-commute, and the u, v spinors commute. In path integral quantisation, write the fields as Grassman spinors, so they anti-commute. But NB, complex conjugation of grassman numbers is defined such that $(\alpha\beta)^* = \beta^*\alpha^*$.

To convert from 4-component notn to 2...:

A 4-comp Grassman spinor ψ_D (attention, ψ_D is often in commuting-spinor notn) can be written as two chiral 2-comp fermions (LH = χ, and RH = $\bar{\eta}$):

$$\psi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\eta}^{\bar{\beta}} \end{pmatrix} \quad , \quad \overline{\psi}_D = \begin{pmatrix} \eta^\alpha & \bar{\chi}_{\bar{\beta}} \end{pmatrix}$$

The 2-comp indices α and $\bar{\beta}$ run from 1..2, and are contracted with the anti-sym epsilon tensor

$$\varepsilon_{\bar{\alpha}\bar{\beta}} = \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon^{\bar{\alpha}\bar{\beta}} = \varepsilon_{\alpha\beta} = -\varepsilon^{\alpha\beta}$$

NB sign flip in going from dotted to undotted (barred in my latex) indices.
Undotted indices are always contracted up-down:

$$\chi\rho = \chi^\alpha \rho_\alpha = \varepsilon^{\alpha\beta}\chi_\beta\rho_\alpha = -\rho_\alpha\chi^\alpha = \rho^\alpha\chi_\alpha$$

and dotted indices down-up, and the ε flips sign in getting bars (sign flip because of up-down vs down-up summing conventions: $\bar{\rho}^{\bar{\beta}} = \bar{\rho}_{\bar{\alpha}}\varepsilon^{\bar{\alpha}\bar{\beta}}$, but $\bar{\rho}^{\bar{\beta}} = (\rho^\beta)^\dagger = (\rho_\alpha\varepsilon^{\beta\alpha})^\dagger$):

$$\begin{aligned} (\eta\rho)^* &= (\eta\rho)^\dagger = (\varepsilon^{\alpha\beta}\eta_\alpha\rho_\beta)^* = (-\varepsilon^{\bar{\alpha}\bar{\beta}})\bar{\rho}_{\bar{\beta}}\bar{\eta}_{\bar{\alpha}} \\ &= \bar{\rho}_{\bar{\alpha}}\bar{\eta}^{\bar{\alpha}} \end{aligned}$$

So, eg

$$\chi_\alpha = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \quad , \quad \chi^\alpha = \begin{pmatrix} \chi_2 & -\chi_1 \end{pmatrix} \quad , \quad \bar{\chi}_{\bar{\alpha}} = \begin{pmatrix} \chi_1^* & \chi_2^* \end{pmatrix} \quad , \quad \bar{\chi}^{\bar{\alpha}} = \begin{pmatrix} \chi_2^* \\ -\chi_1^* \end{pmatrix}$$

In practise, there is a -ve sign from interchanging fermion fields in an operator, but not when you take cc of the op.

trick: the charge conjugate of ψ_L is right-handed

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad , \quad \{\gamma^\alpha\} = \left\{ \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} \right\}$$

$$\{\sigma_i\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\psi^c = -i\gamma_0\gamma_2\bar{\psi}^T = -i\gamma_0\gamma_2\gamma_0\psi^* = i\gamma_2^*\psi^* = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \begin{pmatrix} \psi_L^* \end{pmatrix} \\ \begin{pmatrix} \psi_R^* \end{pmatrix} \end{pmatrix}$$

$$(\psi_L)^c = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \begin{pmatrix} \psi_L^* \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \psi_L^* \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \\ \begin{pmatrix} -i\sigma_2\psi_L^* \end{pmatrix} \end{pmatrix}$$

...useful to:

1. write a 2 component/chiral fermion (*eg* ν_L) in four components
2. write a “Majorana” mass term $m\overline{(\nu_L)^c}\nu_L + h; c;$, for a chiral 2-component fermion
3. construct Lorentz invariant interactions

Diverse notation...

notn: $g \sim 2/3$, $v \simeq 174$ GeV, $m_W \simeq gv/\sqrt{2}$,

$$\frac{g^2}{2m_W^2} = \frac{4G_F}{\sqrt{2}} = \frac{1}{v^2}$$

0.2 References

Field Theory

• Ramond,*FT: A modern primer*

skeletal, linear, almost everything you need to know. I learned FT from this book.

• Peskin + Schroeder,*An intro to QFT*

Almost everything is there, reach it by going round in circles. I love this book.

• Cheng + Li, *Gauge Theory of elem. part. phys.*

Most of SM pheno. SM Feynman rules in the Appendix. I don't follow a good fraction of their derivations.

• Wess and Bagger, *Supersymmetry and Supergravity*

2 component fermions without typos

The Standard Model

Altarelli (summer school lectures, Les Houches '90, ...)

Neutrino masses

Peccei, Gonzalez-Garcia,

Leptoquarks

LFV

Kuno and Okada

EFT

• Georgi, Ann. Rev. Part. Nucl. Phys (words, intuition);

NPB 361[1991) 339 (clear matching formalism, in favour of on-shell matching)

• Simma, why can use the EoM to reduce the operator basis ⇔ on-shell EFT.

0.3 The lepton sector of the SM

lepton particle content:
The SM contains massive charged leptons (e, μ, τ) who leave tracks in detectors (due to electromagnetic interactions). They also have "weak" interactions (eg, participate in β decay).

The SM also contains neutrinos = neutral massless leptons which only interact via weak interactions (eg β decay). There are three (2-comp) neutrinos who contribute to the invisible decays of the $Z \to \nu\bar{\nu}$.

lepton interactions:
Weak interactions are "maximally parity-violating", that is, they only involve χ_Ls. And they can change charged leptons into neutrinos (so a boson-fermion1-fermion2 interaction; more complicated than boson-fermion1-fermion1 of QED)

The (dimensionless) coupling constants for weak and em interactions are $\mathcal{O}(1)$. The interactions can be described as gauge interactions $\Leftrightarrow$ via a covariant derivative. But for the weak interactions, the gauge bosons are massive, so the low-energy interaction is "short-range" = a contact interaction.

Take **one generation** = the electron and one chiral neutrino, decomposed as:

$$\ell_{eL} \equiv \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \quad , \quad \text{and } e_R$$

and give a different covariant derivative to L and R fields.

$$i\,\overline{\ell_L}\,\gamma^\mu \mathbf{D}_\mu\,\ell_L + i\,\overline{e_R}\,\gamma^\mu \mathbf{D}_\mu\,e_R + \text{h.c.} \tag{0.3}$$

The weak interactions are "non-abelian" (covar derivative is a matrix,so have desired $We\nu$ vertices; will also have self-interactions of WWZ!). The U(1) is "hypercharge", not yet electromagnetism...Electroweak interactions are well known (low E precision, LEP). With U(1) charge of ℓ = -1/2:

$$i\,\overline{\ell_L}\,\gamma^\mu \mathbf{D}_\mu\,\ell_L \;=\; i\,\overline{\ell_L}^T\,\gamma^\mu(\partial_\mu + i\frac{g}{2}\sigma^a W^a_\mu - ig'\frac{1}{2}B_\mu)\,\ell_L$$

so the boson-fermion-fermion interactions of eqn (0.3) are (ie, drop the partial derivatives)

$$... - i\,\frac{g}{2}\left(\,\overline{\nu_L}\;\;\overline{e_L}\,\right)\gamma^\mu\left(\left(\begin{array}{cc} W^0_\mu & \sqrt{2}\,W^+_\mu \\ \sqrt{2}\,W^-_\mu & -W^0_\mu \end{array}\right) - \tan\theta_W B_\mu\right)\left(\begin{array}{c}\nu_L \\ e_L\end{array}\right) + ig\tan\theta_W\overline{e_R}\gamma^\mu B_\mu e_R$$

lepton masses

We would also like to write a mass term for the electron: $m_e(\overline{e_R}e_L + \overline{e_L}e_R)$, but this is not gauge invariant...so we introduce the Higgs doublet, who gets a vev

$$-Y_e(\overline{\nu_L},\overline{e_L})\left(\begin{array}{c}\Phi_+ \\ \Phi_0\end{array}\right)e_R \;+\; \text{h.c} \longrightarrow \langle\Phi_0\rangle = v \longrightarrow -Y_e\overline{e_L}ve_R + h.c. = -m_e(\overline{e_L}e_R + \overline{e_R}e_L)$$

The sign is chosen such that $\mathcal{L} \sim \overline{\psi}(i\partial\!\!\!/ \;- m)\psi$, which gives $(p\!\!\!/ \;- m)u(p)e^{-ip\cdot x}$ for spinors.

The Higgs vev breaks most of the gauge interactions — all but a residual U(1) for QED. This also gives the gauge bosons masses.

Three generations

Data says there are three "générations" of leptons (same gauge intéractions, différent masses):

$$\{\ell_{aL}\} \equiv \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \ , \ \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix} \ , \ \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \quad \text{et } \{e_{aR}\} = e_R, \mu_R, \tau_R,$$

with charged lepton masses: $m_e \sim .511$ MeV, $m_\mu \sim 105$ MeV et $m_\tau \sim 1.7$ GeV.

Writing the leptons ("vectors" in génération space) in an arbitrary basis (*not* mass eigenstates):

$$i\,\overline{\ell_L}^b \gamma^\mu \mathbf{D}_\mu \, \ell_L^b + i\,\overline{e_{Ra}}\,\gamma^\mu \mathbf{D}_\mu \, e_{Ra} - \ [Y_e]_{ab} \overline{\ell}_L^a \, \Phi \, e_R^b + h.c.$$

$\Leftrightarrow$ arbitrary Yukawa matrix, $N_g \times N_g$ for the charged leptons.
After the Higgs gets a vev, obtain a charged lepton mass term

$$- \, v[Y_e]_{ab} \overline{e}_L^a \, \Phi \, e_R^b + h.c.$$

Can diagonalise by unitary transformations on the left and right:

$$V_{Le}[\mathbf{Y_e}]V_{Re}^\dagger = D_{Y_e} \qquad V_{L\nu}[\mathbf{Y}_\nu]V_{R\nu}^\dagger = D_{Y_\nu}$$

- my index order is LR
- obtain V_L, V_R by diagonalising the hermitian matrices: $[\mathbf{Y_e}][\mathbf{Y_e}]^\dagger = V_L^\dagger D_{Y_e}^2 V_L$, $[\mathbf{Y_e}]^\dagger [\mathbf{Y_e}] = V_R^\dagger D_{Y_e}^2 V_R$.

basis choice: with canonical kinetic terms (always possible) *the only* intéraction in $\mathcal{L}$ which has an eigenbasis in the e_R flavour space is $[\mathbf{Y_e}]^\dagger[\mathbf{Y_e}]$. And the intéraction dans $\mathcal{L}$ with an eigenbasis in ℓ_L space is $[\mathbf{Y_e}][\mathbf{Y_e}]^\dagger$.
Donc work in the eigenbasis of $[\mathbf{Y_e}]$. Called the "flavour basis". (stay there even when there are ν masses —this is different from quarks!).

tangent—canonical kinetic terms?

Quoi empeche : (its gauge invariant)

$$(\overline{\nu_{eL}}, \overline{e_L}) \not{D} \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}$$

Can/should write the Lagrangien:

$$iZ_{bc}\, \overline{\ell_L}^{b'}\, \gamma^\mu \mathbf{D}_\mu\, \ell_L^{c'} - [\tilde{Y}_e]_{bd} \overline{\ell}_L'^{b}\, \Phi\, e_R^d + h.c.$$

But its equivalent: to return to the canonical Lagrangien, diagonalise Z

$$\overline{\ell_L'^b} \mathbf{Z}_{bc} \not{D}\ \ell_L'^c = \overline{\ell_L^{b''}} \mathbf{D}_{\mathbf{Z}bb} \not{D}\ \ell_L^{b''} = \overline{\ell_L}^b \not{D}\ \ell_L^b$$

and adsorb the eigenvalues of Z into the fields: $\ell_L^b = \sqrt{z^b}\ell_L^{b''}$.
So with a redefinition of matrice de Yukawa: $\mathbf{Y_e} = \mathbf{D_Z^{-1/2}}\widetilde{\mathbf{Y}}_\mathbf{e}$, obtain canonical kinetic terms

$$\mathcal{L} \;=\; i\, \overline{\ell_L^b}^T \not{D}\ \ell_L^b + i\, \overline{e_R^a} \not{D}\ e_R^a + -\{\, (\overline{\ell_L^b}\, [\mathbf{Y_e}]_{bb}\ \Phi^c) e_R^b + \text{h.c.}\}$$

$\Rightarrow$ in bottom-up pheno, can always take canonical kinetic terms.
In top-doqn model-building, need to compute Z!

0.4 Some phenomenology — current bounds

some processes	current sensitivities
$BR(\mu \to e\gamma)$	$< 5.7 \times 10^{-13}$
$BR(\mu \to e\bar{e}e)$	$< 1.0 \times 10^{-12}$
$\frac{\sigma(\mu+Au \to e+Au)}{\sigma(\mu \text{ capture})}$	$< 7 \times 10^{-13}$
$BR(\tau \to \ell\gamma)$	$< 3.3, 4.4 \times 10^{-8}$
$BR(\tau \to 3\ell)$	$< 1.5 - 2.7 \times 10^{-8}$
$BR(\tau \to e\phi)$	$< 3.1 \times 10^{-8}$
$BR(\tau \to \ell + X_{m \lesssim m_\pi})$	$< 2.7 - 5 \times 10^{-3}$
$BR(\overline{K_L^0} \to \mu\bar{e})$	$< 4.7 \times 10^{-12}$
$BR(K^+ \to \pi^+\bar{\nu}\nu)$	$= 1.7 \pm 1.1 \times 10^{-10}$
$BR(\overline{K^+} \to \pi^+ X_{m\sim 0})$	$< 5.9 \times 10^{-11}$
$BR(B^+ \to K^+\tau\bar{\mu})$	$< 7.7 \times 10^{-5}$

$$BR(\mu \to e\gamma) \equiv \frac{\Gamma(\mu \to e\gamma)}{\Gamma_{tot}(\mu \to ...)}$$

NB: in these lectures, I assume LFV proceeds via contact interactions involving SM particles — so I neglect possible light *new* particles

0.4.1 Muon decays

The muon μ ($m = 0.106$ GeV) decays almost always $\boldsymbol{\mu \to e\overline{\nu}_e\nu_\mu}$ via the weak interactions, with a lifetime of 2.2×10^{-6} seconds, or decay rate (see problem)

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

In these lectures, we will be much interested in the tightly constrained decay $\mu \to e\gamma$. However, it cannot proceed via a renormalisable gauge interaction $\overline{e}\gamma^\alpha A_\alpha \mu$ — its forbidden by the Ward Identity of QED. We can try to calculate the form of a local "contact interaction" which could describe this decay, and the associated rate.

We want the decay of $\boldsymbol{\mu(p_\mu) \to e(p_e)\gamma(q)}$ with all the legs on-shell. The form of the matrix element can be restricted by Lorentz and gauge invariance:

1. The amplitude should be a Lorentz 4-vector $\mathcal{J}_\alpha$ contracted with the photon polarisation vector ε^α: $\mathcal{M} \propto \varepsilon_\alpha \mathcal{J}^\alpha$. A generic vector-fermion-fermion vertex can be expanded on a complete basis of 4*4 matrices, which can be sanwiched between the $\overline{u_e}$ and u_μ spinors. A convenient basis is $\{\Gamma_X\} = \{1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$. So:

$$\mathcal{M} = \varepsilon_\alpha \mathcal{J}^\alpha = \varepsilon_\alpha \overline{u_e}[(S + P\gamma_5)q^\alpha + V\gamma^\alpha + A\gamma^\alpha\gamma_5 + Tq_\nu\sigma^{\alpha\nu}]u_\mu$$

   where S, P, V, A and T are unknown couplings. (why not a term $\propto p^\mu$? I think can get rid of via Gordon?)

2. the on-shell photon has two transverse degrees of freedom, and an arbitrary (gauge-dep) coefficient of q_α. So the component of $\mathcal{J}^\alpha$ along q^α cannot contribute (equivalently, $\varepsilon_T \cdot q_\alpha = 0$, and $q^2 = 0$.

3. Then, we saw that non-canonical kinetic terms of the form $\overline{e}\not{D}\ \mu$ could be rotated away. So we anticipate that also V and maybe A will not contribute. This can be seen using QED gauge invariance in the form of the Ward identity (P+S 7.64)

$$q_\alpha \mathcal{J}^\alpha \;=\; 0 = \overline{u_e}[V\not{q}\ + A\not{q}\ \gamma_5]u_\mu$$

   Using the Eqns of motion for the external fermions

$$(\not{p}_\mu - m_\mu)u_\mu = 0 \qquad , \qquad \overline{u_e}(\not{p}_\mu - m_e) = 0$$

   one obtains $A = V = 0$.

So the decay is via the dipole part of the amplitude $\propto T$.

0.5 Problems about SM lepton decays

0.5.1 muon decay

1. Draw the Feynman diagram for the decay $\mu(P) \to \nu_\mu(k_2) \ + \ e(p) \ + \ \overline{\nu_e}(k_1)$.

2. Calculate the matrix element $\mathcal{M}$ (SM Feynman rules in the Appendix of Cheng + Li). Can you approximate the propagator of the $W \propto g^{\mu\nu}/m_W^2$? . Calculate $|\mathcal{M}|^2$ doing the muon spin sum. Use

$$\frac{4G_F}{\sqrt{2}} = \frac{g^2}{2m_W^2}$$

. You should obtain

$$\begin{aligned} |\mathcal{M}|^2 &= 2\left(\frac{4G_F}{\sqrt{2}}\right)^2 [P_\mu k_{2\nu} + k_{2\mu}P_\nu - P \cdot k_2 g_{\mu\nu} - i\epsilon_{\alpha\nu\beta\mu}P^\alpha k_2^\beta][p^\mu k_1^\nu + k_1^\mu p^\nu - p \cdot k_1 g^{\mu\nu} - i\epsilon^{\rho\nu\sigma\mu}k_{1,\rho}p_\sigma] \\ &= 8\left(\frac{4G_F}{\sqrt{2}}\right)^2 P \cdot k_1 k_2 \cdot p \end{aligned} \tag{0.4}$$

3. The decay rate is obtained by integrating

$$d\Gamma = \frac{(2\pi)^4}{2M}|\mathcal{M}|^2 d\Phi_3(P; p, k_1, k_2) \tag{0.5}$$

- Check that you can get (0.13) from (0.7).
- Neglect the electron and neutrino masses, calculate in the muon rest frame, and express $k_2 \cdot p$, $|\vec{p}_1^*|$ and k_1^0 as functions of m_{12}, to obtain

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3} \tag{0.6}$$

The integration limits for m_{12} are discussed in the PDB, fig 39.3 of the "Kinematics" review (PDB= http://hepdata.cedar.ac.uk/lbl/).

0.5.2 Problem: Calculate $\mu \to e\gamma$

For a matrix element

$$\mathcal{M} = T\varepsilon_\alpha \overline{u_e}[q_\rho \sigma^{\alpha\rho}]u_\mu$$

calculate the decay rate for $\mu \to e\gamma$:

$$\Gamma(\mu \to e\gamma) = |T|^2 \frac{m_\mu^3}{4\pi}$$

Notice that:

- phase space is $\sim 4\pi/(16\pi^3)$ per particle (see discussion after $\mu \to e\nu\bar{\nu}$ problem), so this two-body decay *gains* a factor $4\pi^2$, with respect to the tree weak decay of the muon!
- T must have inverse mass dimension.
- In addition, its difficult to draw a diagram for $\mu \to e\gamma$ at tree level, so probably $T \propto 1/(16\pi^2)$.
- And since a photon is emitted, need $T \propto e$.

So one might guess

$$\Gamma(\mu \to e\gamma) \lesssim \frac{\alpha_e m_\mu^3}{256\pi^4 \Lambda_{NP}^2} \quad \Rightarrow \quad BR(\mu \to e\gamma) \lesssim \frac{3\alpha_e}{4\pi} \frac{1}{G_F^2 m_\mu^2 \Lambda_{NP}^2} \leq 5.7 \times 10^{-13}$$

We will see that many BSM models predict much smaller rates than this. Fortunately.

0.5.3 tau decays

The masses of SM fermions are $m_t = 175$ GeV, $m_b = 4.2$ GeV, $m_c = 1.3$ GeV, $m_s = 0.095$ GeV, $m_{u,d} \simeq 5 - 10$ MeV, $m_\tau = 1.77$ GeV, $m_\mu = 0.106$ GeV, $m_e = 0.511$ MeV.

The tau τ ($m = 1.8$ GeV) also decays via the weak interactions (W exchange) with a lifetime $\sim 3 \times 10^{-13}$seconds. It can decay to $e\overline{\nu}_e\nu_\mu$ and $\mu\overline{\nu}_\mu\nu_\tau$ with branching ratios $\sim 1/5$ (respectively 17.8, 17.4 %). Why this BR? (see problem).

An (exptal) difficulty with LFV τ decays, compared to μ decays, is the short τ lifetime. And one "loses" the BR factor of 1/5.

But τs are interesting, because they are 3rd generation (more sensitive to New Physics with hierarchical couplings?), and can decay to quarks (probes different NP from μ).

1. Assuming that the τ decays via weak interactions, like the μ, what are the final state allowed by kinematics, charge conservation, and the interactions of the W?
   Its useful to recall that the lightest charmed mesons are 1.87 GeV ($D^0, D^\pm$).

2. Suppose that the weak decay is a contact interaction, and that hadronisation happens afterwards — what do you estimate for the branching ratio for $\tau \to \mu\nu\overline{\nu}$?

3. Given the formula (0.6) for muon decay, estimate the lifetime of the τ. Do you obtain 3×10^{-13} seconds?

0.5.4 meson decays to final states with leptons

...discuss this in leptoquark chapter...

Appendix to problems: phase space

Phase space for n particules of 4-momentum $p_i = E_i\vec{p_i})$, produced from an initial state of four-momentum P with $P^2 = M^2$, can be written (Particle Data Book, revue "Kinematics", eqn 39.11)

$$d\Phi_n(P;p_1..p_n) = \delta^4(P - \sum_i p_i)\Pi_i \frac{d^3p_i}{2E_i(2\pi)^3} \tag{0.7}$$

(its Lorentz invariant). The **2 body** expression is:

$$\begin{aligned}
d\Phi_2(P;p_1,p_2) &= \delta^4(P - p_1 - p_2)\frac{d^3p_1}{2E_1(2\pi)^3}\frac{d^3p_2}{2E_2(2\pi)^3} \\
&= \delta^4(P - p_1 - p_2)\frac{d^4p_1}{(2\pi)^3}\delta(p_1^2 - m_1^2)\theta(E_1)\frac{d^3p_2}{2E_2(2\pi)^3} \\
&= \frac{1}{(2\pi)^3}\delta((P - p_2)^2 - m_1^2)\frac{d^3p_2}{2E_2(2\pi)^3} \\
&= \frac{1}{(2\pi)^3}\delta(M^2 + m_2^2 - 2ME_2 - m_1^2)\frac{|\vec{p_2}|^2 dp_2 d\Omega_2}{2E_2(2\pi)^3} \\
&= \frac{1}{(2\pi)^3}\delta(M^2 + m_2^2 - 2ME_2 - m_1^2)\frac{|\vec{p_2}| dE_2 d\Omega_2}{2(2\pi)^3} \\
&= \frac{|\vec{p_2}| d\Omega_2}{4M(2\pi)^6}|_{E_2=(M^2+m_2^2-m_1^2)/(2M_2)}
\end{aligned} \tag{0.8}$$

On the 3rd line, supposed $P^2 = M^2$ and the initial state rest frame, so $P = (M, 0, 0, 0)$ and $P \cdot p_2 = ME_2$. For the kinematics:

$$E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M_2} \quad , \quad E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M_2} \tag{0.9}$$

and.... (PTO)

... $|\vec{p_1}| = |\vec{p_2}| = \sqrt{E_1^2 - m_1^2}$, so

$$
\begin{aligned}
|\vec{p_1}| &= \frac{M}{2}\sqrt{(1+y-x)^2 - 4y} = \frac{M}{2}\sqrt{1-2(x+y)+(y-x)^2} \\
&= \frac{1}{2M}\sqrt{(M^2+m_1^2-m_2^2)^2 - 4M^2m_1^2} = \frac{1}{2M}\sqrt{[M^2-(m_2-m_1)^2][M^2-(m_2+m_1)^2]}
\end{aligned}
\tag{0.10}
$$

for $x = m_1^2/M^2$, $y = m_2^2/M^2$.

This result also uses a useful identity of δ "functions":

$$
\int dx f(x)\delta(ax-b) = \frac{1}{a}\int d(ax) f(x)\delta(ax-b) = \frac{1}{a}f|_{x=b/a}
$$

To obtain a useful representation of **3 body** phase space, start by checking

$$
d\Phi_n(P;p_1..p_n) = d\Phi_{n-j+1}(P;p_{j+1}..p_n,q)(2\pi)^3 dq^2 d\Phi_j(q;p_1..p_j) \tag{0.11}
$$

by using $dq^2 \equiv d(q_0^2 - \vec{q}^2) = 2q_o dq_0$, and

$$
\int d^4q\,\delta^4(P - \sum_{i=j+1}^{n} p_i - q)\delta^4(q - \sum_{i=1}^{i=j} p_i) = \delta^4(P - \sum_{i=1}^{n} p_i) \tag{0.12}
$$

Then, for $q^2 = (p_1+p_2)^2 \equiv m_{12}^2$, one can obtain (voir PDB, 39.19)

$$
\begin{aligned}
d\Phi_3(P;p_1,p_2,p_3) &= d\Phi_2(P;p_3,q)(2\pi)^3 dm_{12}^2 d\Phi_2(q;p_1,p_2) \\
&= \frac{dm_{12}}{8M(2\pi)^9}\ |\vec{p_3}|d\Omega_3\ |\vec{p_2}^{\,*}|d\Omega_2^*
\end{aligned}
\tag{0.13}
$$

where the starred coordinates are in the rest frame of the 12 system (that is, $q = (m_{12}, \vec{0})$), and

$$
2M|\vec{p_3}| = \sqrt{[M^2-(m_3-m_{12})^2][M^2-(m_3+m_{12})^2]} \qquad 2m_{12}|\vec{p_2}^{\,*}| = \sqrt{[m_{12}^2-(m_2-m_1)^2][m_{12}^2-(m_2+m_1)^2]} \tag{0.14}
$$