3 Dirac Neutrino Masses

- add singlet neutrinos to SM, and conserve lepton number
- Dirac m_ν (like quarks) and PMNS matrix
- FCNC at loop, suppressed by GIM mechanism
- compute $\mu \to e\gamma$ in SM...
- **...and identify as a dimension 5 effective operator in QED!**
  "integrating out" the $W(Z, h)$, in loops"

Why are Dirac masses interesting?

- minimal extension of SM giving neutrino masses
- it works for the quarks
- we don't understand Yukawas, so maybe its ok that Y_ν is small
- there are two approaches to adding New Particles to the SM, such that one has not yet detected them:
  - they are heavy (not produceable in current expts)
    could have $\mathcal{O}(1)$ couplings, can be decribed using EFT
  - they are light and feebly coupled

  Dirac neutrinos are an example of the second approach. Maybe its worth exploring...

3.1 The lepton Lagrangian, with gauge-singlet fermions and Yukawas

1) Add to the SM some fermions *without* gauge interactions (refered to as ν_Rs, or singlet neutrinos).
2) Attribute to them *lepton number* L, and impose that L is conserved in the Lagrangian.
Then the most general renormalisable Lagrangian for one generation is the SM + the second line

$$i\,\overline{\ell_L}\,\gamma^\mu \mathbf{D}_\mu\,\ell_L + i\,\overline{e_R}\,\gamma^\mu \mathbf{D}_\mu\,e_R - \; Y_e(\overline{\nu_L},\overline{e_L}) \begin{pmatrix} \Phi_+ \\ \Phi_0 \end{pmatrix} e_R \; + \; \text{h.c.}$$

$$+i\,\overline{\nu_R}\,\gamma^\mu \partial_\mu\,\nu_R + Y_\nu(\overline{\nu_L},\overline{e_L}) \begin{pmatrix} -\Phi_0^* \\ \Phi_- \end{pmatrix} \nu_R \; + \; \text{h.c.}$$

(sign choices to get free EoM $(\not p\; - m)\psi = 0$). Giving a vev to the neutral Higgs field $\langle\Phi_0\rangle = v \simeq 174$ GeV, one obtains "Dirac" masses (ie, connecting two independent chiral fermions) for the electron and neutrino

$$-Y_e v \overline{e_L} e_R + h.c. - Y_\nu v \overline{\nu_L} \nu_R + h.c.$$

So notice that $Y_\nu \ll Y_e \sim 3\times 10^{-6}$, to obtain $m_\nu \lesssim$ eV $\ll 0.5\times 10^6 eV$.

Now, consider 2 generations in an arbitrary basis. Canonical kinetic terms are invariant under flavour basis transformations of gauge mutiplets, so ignore kinetic tersm for the moment. Yukawa interactions:

$$-\,[Y_e]_{ab}\overline{\ell}_L^a\,\Phi\,e_R^b + \;[Y_\nu]_{ab}\overline{\ell}_L^a\,\widetilde{\Phi}\,\nu_R^b + h.c.$$

TWO *Yukawa* matrices, *arbitrary* in flavour space (not gauge SU(2)!). Can diagonalise both by indep transformations on the left and right:

$$V_{Le}[\mathbf{Y_e}]V_{Re}^\dagger = D_{Y_e} \qquad V_{L\nu}[\mathbf{Y}_\nu]V_{R\nu}^\dagger = D_{Y_\nu}$$

(Suppose canonical kinetic terms, and) diagonalise:

$$V_{Le}[\mathbf{Y_e}]V_{Re}^\dagger = D_{Y_e} \qquad V_{L\nu}[\mathbf{Y}_\nu]V_{R\nu}^\dagger = D_{Y_\nu}$$

Now identify eigenbases of $\mathcal{L}$:

- In e_{Ra} flavour space, only $[\mathbf{Y_e}]^\dagger[\mathbf{Y_e}]$ has an eigenbasis.
  This is charged lepton flavour basis, always work there.
- In ν_{Ra} flavour space, only $[\mathbf{Y}_\nu]^\dagger[\mathbf{Y}_\nu]$ has an eigenbasis.
  This is the neutrino mass basis. If/when need a basis for ν_Rs, use this one.
- TWO eigenbases in doublet ℓ_L flavour space:
  $[\mathbf{Y_e}][\mathbf{Y_e}]^\dagger \Rightarrow$ charged lepton mass basis, or "flavour basis", *usually use this basis*
  $[\mathbf{Y}_\nu][\mathbf{Y}_\nu]^\dagger \Rightarrow$ "neutrino mass basis"

In the flavour basis, can write the Lagrangian:

$$i\,\overline{\ell_L}^{bT}\,\gamma^\mu \mathbf{D}_\mu\,\ell_L^b + i\,\overline{e_R}^a\,\gamma^\mu \mathbf{D}_\mu\,e_R^a + i\,\overline{\nu_R}^b\,\gamma^\mu\partial_\mu\,\nu_R^b - \,[D_{Ye}]_{bb}\overline{\ell}_L^b\,\Phi\,e_R^b + \,\overline{\ell}_L^d V_{Le}V_{L\nu}^\dagger[D_{Y\nu}]_{dd}\,\widetilde{\Phi}\,\nu_R^d + h.c.$$

and define $V^\dagger = V_{Le}V_{L\nu}^\dagger$ to be the lepton mixing matrix (or PMNS matrix).

3.2 Where does that mixing matrix appear?

Suppose two lepton generations, Dirac masses for neutrinos and charged leptons. Work in the usual perspective that mass eigenstates propagate (also foir neutrinos). Choose the mass eigenstate bases for $\{e_R^\alpha\} = (\mu_R, e_R)$, and $\{\nu_R^i\} = (\nu_R^2, \nu_R^1)$. For the ℓ^a flavour space, take the flavour basis, and write explicitly $\nu_{Le} = V_{ei}\nu_L^i$

$$\{\ell_L^\alpha\} \equiv \begin{pmatrix} V_{ei}\nu_L^i \\ e_L \end{pmatrix} \quad , \quad \begin{pmatrix} V_{\mu j}\nu_L^j \\ \mu_L \end{pmatrix}$$

and now put this is the Lagrangian:

$$i\,(V_{ej}^*\overline{\nu_L^j}\ \ \overline{e_L})\,\gamma^\mu \mathbf{D}_\mu \begin{pmatrix} V_{ei}\nu_L^i \\ e_L \end{pmatrix} + i\,(V_{\mu j}^*\overline{\nu_L^j}\ \ \overline{\mu_L})\,\gamma^\mu \mathbf{D}_\mu \begin{pmatrix} V_{\mu i}\nu_L^i \\ \mu_L \end{pmatrix} + i\,\overline{e_R}\,\gamma^\mu \mathbf{D}_\mu\, e_R + i\,\overline{\mu_R}\,\gamma^\mu \mathbf{D}_\mu\, \mu_R$$
$$-\, y_e(\overline{\nu_L^e}\ \overline{e_L}) \begin{pmatrix} \Phi_+ \\ \Phi_0 \end{pmatrix} e_R - y_\mu(\overline{\nu_L^\mu}\ \overline{\mu_L}) \begin{pmatrix} \Phi_+ \\ \Phi_0 \end{pmatrix} \mu_R - y_{\nu 1}(\overline{\nu_L^1}\ ....) \begin{pmatrix} \Phi_0 \\ -\Phi_- \end{pmatrix} \nu_R^1 - y_{\nu 2}(\overline{\nu_L^2}, ...) \begin{pmatrix} \Phi_0 \\ -\Phi_- \end{pmatrix} \nu_R^2 + \text{h.c.}$$

and discover that the mixing matrix appears at the W^+ vertices:

$$-i\,\frac{g}{2}\,(V_{ej}^*\overline{\nu_L^j}\ \ \overline{e_L})\gamma^\mu \left(\begin{pmatrix} W_\mu^0 & \sqrt{2}\,W_\mu^+ \\ \sqrt{2}\,W_\mu^- & -W_\mu^0 \end{pmatrix} - \tan\theta_W B_\mu \right) \begin{pmatrix} V_{ei}\overline{\nu_L^i} \\ e_L \end{pmatrix}$$

So have the familiar (from quark sector) Feynman rule:

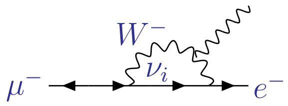
$$-i\,\frac{gV_{ej}^*}{\sqrt{2}}\overline{\nu_L^j}\gamma^\mu W_\mu^+ e_L$$

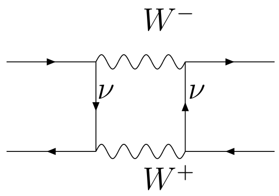
Notice that the Z vertex is diagonal, both for charged leptons, and mass eingenstate neutrinos:

$$\propto \sum_\alpha -i\,\frac{g}{2} V_{\alpha j}^* \overline{\nu_L^j}\gamma^\mu Z_\mu^+ V_{\alpha i}\nu_L^i = \delta_{ij}\frac{g}{2}\overline{\nu_L^i}\gamma^\mu Z_\mu^+ \nu_L^j$$

This is the lepton version of the famous absence of tree-level Flavour-Changing-Neutral-Currents (in the quark sector of the SM). It means that with Dirac neutrino masses, LFV can only arise in loops with W exchange.

3.3 Some LFV loops





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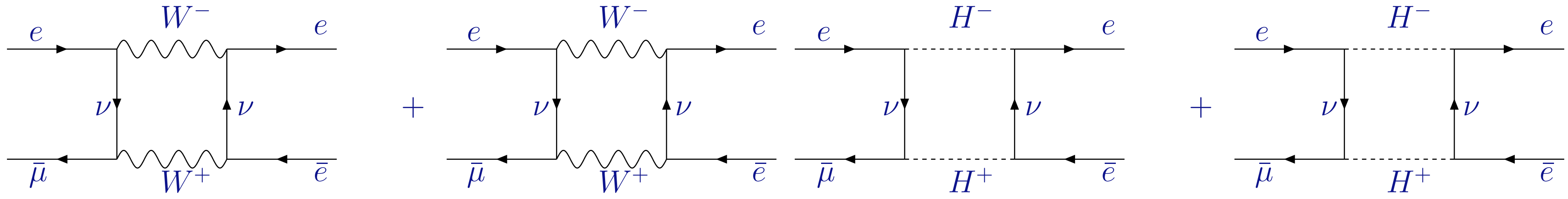
1. Want at least two charged lepton external legs, of different flavour.
2. flavour changes at W vertices, so can draw a box? Or a triangle?

Start with the box(Feynman 'tHooft gauge): but there are penguins too, and always need all the diagrams!



Write amplitude, zero external momenta, neglecting neutrino masses+ Yukawas(so drop the goldstone):

$$\begin{aligned}\mathcal{M} &\propto 2(V_{e1}V^*_{\mu 1} + V_{e2}V^*_{\mu 2})\frac{g^4}{4}\int \frac{d^4p}{(2\pi)^4}[\gamma^\mu P_L \frac{\not{p}}{p^2}\gamma^\nu P_L]\frac{i}{p^2-m_W^2}\frac{i}{p^2-m_W^2}[\gamma_\mu P_L \frac{\not{p}}{p^2}\gamma_\nu P_L] \\ &\propto 0 \times \frac{g^2}{16\pi^2}G_F\end{aligned}$$

Power-counting finite, so no divergence—but zero by unitarity of lepton mixing matrix :(

Keep $m_{\nu 2} \lesssim$ eV (negliger $m_{\nu 1} \ll$) in W diagrams:

$$\begin{aligned}\mathcal{M} &\propto V_{e2}V^*_{\mu 2}\frac{g^4}{4}\int \frac{d^4p}{(2\pi)^4}[\gamma^\mu P_L \frac{\not{p}}{p^2}\left(1+\frac{m_2^2}{p^2}\right)\gamma^\nu P_L]\frac{i}{p^2-m_W^2}\frac{i}{p^2-m_W^2}[\gamma_\mu P_L \frac{\not{p}}{p^2}\gamma_\nu P_L] \\ &\propto V_{e2}V^*_{\mu 2}G_F\frac{g^2}{16\pi^2}\frac{m_{\nu 2}^2}{m_W^2} \lesssim 10^{-25}G_F\end{aligned}$$

(but at $\mathcal{O}(Y_\nu^2)$ should include diagrams with a goldstone!)

GIM mechanism (Glashow Iliapoulos, Maiani), reduces FCNC in the quark sector ($m_t^2/m_W^2 \sim 1$, but $V_{td}, V_{ts} \lesssim 10^{-2}$); renders invisible in the lepton sector for Dirac neutrino masses, even though lepton sector mixing angles are $\mathcal{O}(1)$.

3.4 $\mu \to e\gamma$ in presence of m_ν

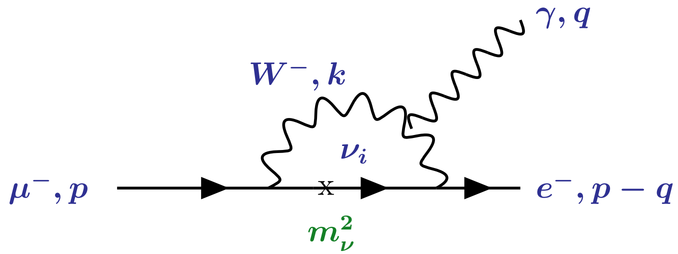


Diagram called a "penguin": replace the W propagator by a blob (the penguin's head), then the external fermions are the wings, and the neutrino is the contour of the body. This penguin has a hair and no feet.

Matrix element, in unitary gauge, with ε the photon polarisation vector, and using the $\gamma(-q)W^{+\alpha}W^{-\beta}(-k)$ vertex from C+L, with momenta ingoing, $= -ie[(-2q-k)_\beta g_{\gamma\alpha} + (-2k-q)_\gamma g_{\beta\alpha} + (q+2k)_a g_{\gamma\beta}]$:

$$\begin{aligned}
\mathcal{A} &\sim \sum_i U_{ei}U^*_{\mu i} e \frac{g^2 m_\mu}{2} \overline{u_e} \int \frac{d^4k}{(2\pi)^4} i\gamma^\mu P_L \frac{(-i)^2 i(\not p + \not k)}{(k^2 - m_W^2)((k+p)^2 - m_i^2)((k+q)^2 - m_W^2)} i\gamma^\nu P_L u_\mu \times \\
&\qquad [g^{\nu\beta} + k^\nu k^\beta / m_W^2][g^{\mu\alpha} + (k+q)^\mu (k+q)^\alpha / m_W^2] - i[(-2q-k)_\beta g_{\gamma\alpha} + (-2k-q)_\gamma g_{\beta\alpha} + (q+2k)_a g_{\gamma\beta}]\varepsilon^\gamma \\
&\sim 0 \times \text{divergent} + \mathcal{O}\left(\frac{m_\nu^2}{m_W^2}\right)
\end{aligned}$$

The divergence is not a surprise; the diagram renormalises the γee vertex (with many other diagrams...). The multiplicative GIM suppression is there.

Suppose we want to calculate — **back up a bit:**

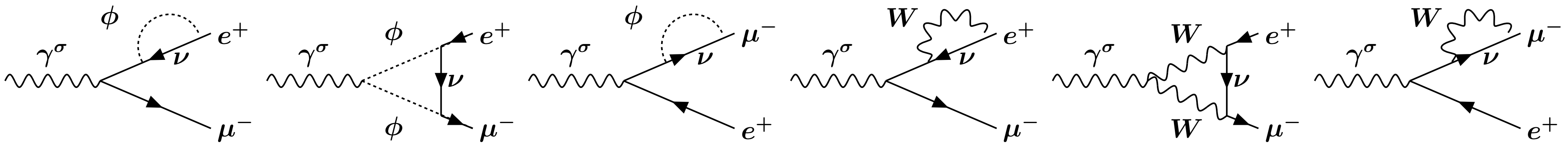
1. what is the complete set of diagrams?
2. what gauge are we computing in? And do we know what to do with γ_5?
3. what are we trying to calculate anyway?

3.5 The divergence

Peskin + Schroeder, eqn 7.46, sec 7.4

...is not without interest. For instance, what if the mixing matrix is not-unitary?Do we need a counter term for the $\gamma\mu e$ vertex? Does that mean there be a renormalisable $\gamma\mu e$ vertex?

1. Need all the diagrams to get correct cancellations arising from gauge symmetries — and don't use unitary gauge for divergent loops (SM Feynman Rules in Appendix of Cheng+Li):



2. (for algebraically challenged people) **Recall also**: Ward Takahashi identities of QED equates the renormalisation factors $\boldsymbol{Z_1 = Z_2}$ of the propagators and of the vertex:

$$Z_1 \overline{\psi} \partial\!\!\!/ \, \psi - ieZ_2 \overline{\psi} A\!\!\!/ \, \psi$$

   This is *not* just the cancellation of the divergences; it is true for finite parts also, to all orders in perturbation theory according to Peskin + Schroeder, around eqn 7.46, and sec 7.4.

3. what to do with $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ in $4 - 2\epsilon$ dimensions?

   tHooft +Veltman, NPB44, 189
   Trueman,hep-ph/9504315
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   - naively use 4d defn + cross your fingers (usually works — see eg Trueman)
   - 'tHooft Veltman: always works. Its more intricate (some additional subtractions in $\overline{MS}$)

Moral: if you find renormalisable flavour-changing vertices for a gauge boson...its very interesting. Check again. If there is a doubtful γ_5, find a collaborator who knows what to do with them.

3.6 A recipe for extracting the dipole part of the amplitude

Recall that the decay $\mu(p) \to e\gamma(q)$ occurs via the dipole part of the amplitude. That is, if the general $\mu e\gamma$ vertex is written

$$\mathcal{M} = \varepsilon_\alpha \overline{u_e}[(S + P\gamma_5)q^\alpha + V\gamma^\alpha + A\gamma^\alpha\gamma_5 + Tq_\nu\sigma^{\alpha\nu}]u_\mu$$

then $\Gamma(\mu \to e\gamma) = |T|^2 m_\mu^3/4\pi$.

However, the one-loop calculation of the $\mu e\gamma$ vertex will also contain the divergence-that-should-cancel, which we are not interested in. So we would like a recipe to extract the dipole coefficient. This is an interesting simplification because the diagrams with loops on external legs do not contribute to the dipole (so we could drop them), but are neccessary to cancel the divergent coefficient of γ^α.

1. we first rewrite the general form of $\mathcal{M}$ using chiral projectors (SM fermions are chiral):

$$\mathcal{M} = \varepsilon_\alpha \overline{u_e}[(S+P)P_R q^\alpha + (S-P)P_L q^\alpha + (V+A)P_R\gamma^\alpha + (V-A)P_L + q_\nu\sigma^{\alpha\nu}(\sigma_L P_L + \sigma_R P_R)]u_\mu$$

   (so what was T is now $\sigma_{L,R}$; this is to match the notn of Lavoura.)

2. then we use the Gordon decomposition

$$\overline{u_e}(p_e)\gamma^\alpha P_L u_\mu(p_\mu) = \frac{1}{m_\mu}\overline{u_e}(p_e)[p_\mu^\alpha - i\sigma^{\alpha\beta}p_{\mu,\beta}]P_R u_\mu(p_\mu) = \frac{1}{m_e}\overline{u_e}(p_e)[p_e^\alpha + i\sigma^{\alpha\beta}p_{e,\beta}]P_L u_\mu(p_\mu)$$

   (show this via $\overline{u_1}(p_1)\gamma^\mu \not{p}_2 u_2(p_2) + \overline{u_1}(p_1)\not{p}_1\gamma^\mu u_2(p) = (m_1 + m_2)\overline{u_1}(p)\gamma^\mu u_2(p-q)$) to rewrite the dipole as

$$i\overline{u_e}q_\beta\sigma^{\alpha\beta}P_L u_\mu = \overline{u_e}[-\gamma^\alpha(m_e P_L + m_\mu P_R) + (2p_\mu^\alpha - q^\alpha)P_L]u_\mu$$

   So the coefficient σ_L will be $-i/2\times$ the coefficient of $\varepsilon \cdot p_\mu$

So we obtain the follwing recipe **To extract the dipole part from the messy amplitudes:**
1) write amplitude in terms of p_μ and q (get rid of p_e)
2) reduce amplitude to having one or less γ matrices
3) retrieve $\varepsilon \cdot p$ terms

The calculation, in a general $R-\xi$ gauge, is in C+L, chapter 13. They find (multiplicative GIM suppression (no leptonic log-GIM because ν not couple to γ ...)

$$\sigma_L \simeq m_\mu G_F \frac{e}{16\pi^2} \frac{U_{\mu i} m_{\nu,i}^2 U_{ei}^*}{m_W^2} \tag{3.46}$$

$$\Rightarrow \Gamma(\mu \to e\gamma) = \frac{|\sigma_L|^2 + |\sigma_R|^2}{16\pi} \qquad , \quad BR(\mu \to e\gamma) \sim \frac{3\alpha_{em}}{16} \left(\frac{m_\nu}{m_W}\right)^4$$

For BSM calculations, see the general formulae of Lavoura, hep-ph/0302221, EPJC29(2003)191.

3.7 Effective operators for the low-energy theory of QED

1. The decay $\mu \to e\gamma$ is the emission of a photon by electrically charged leptons. We should therefore be able to write it as an interaction in QED.

2. So we would like to describe $\mu \to e\gamma$ with a non-renormalisable contact interaction. It should be QED gauge invariant, Lorentz invariant, have μ, e, γ legs, and the correct Lorentz structure. Something like

$$\overline{e}\sigma_{\alpha\beta}\mu F^{\alpha\beta} = 2\overline{e}\sigma_{\alpha\beta}\mu\partial^\alpha A^\beta$$

satisfies these criteria and has mass dimension 5

3. We also saw that, at scales below m_W, the weak interactions induce contact interactions which are invariant under the electroweak gauge symmetry $SU(2) \times U(1)$, not just $U(1)_{em}$. The dipole flips the chirality of the fermion:

$$\overline{e}\sigma^{\alpha\beta}P_R\mu F_{\alpha\beta} = e^\dagger P_L\gamma^0\sigma^{\alpha\beta}\mu F_{\alpha\beta}$$

so a gauge invariant interaction in the SM must be constructed with doublet and singlet lepton fields,and therefore also a Higgs, so the dipole interaction is dimension 6, and can involve singlet muons or electrons. So in the Lagrangian, one could put the operators and coefficients:

$$\frac{\sigma_R}{2v}(\overline{\nu_L e_L})\begin{pmatrix}0\\v\end{pmatrix}\sigma^{\alpha\beta}\mu_R F_{\alpha\beta} + \frac{\sigma_L^*}{2v}(\overline{\nu_L \mu_L})\begin{pmatrix}0\\v\end{pmatrix}\sigma^{\alpha\beta}e_R F_{\alpha\beta} + h.c.$$

(σ_X are defined as coefficient of interaction annihilating a μ_X).